between our and Oden and Key's results may be caused by the distinction of two methods, the Newton-Raphson and the incremental loading used by them.

References

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Technical Comments

Comment on "Perturbation Method of Structural Design Relevant to Holographic Vibration Analysis"

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ALTHOUGH Stetson[1] has referred to an original source[2] for Rayleigh's principle, he has apparently overlooked, as many others have, the original and complete treatment of perturbation theory, also given by Rayleigh in that same source. In fact, except for one small, but sometimes significant, difference, Stetson's principal results, namely Eq. (21) for the changes in frequencies and Eq. (30) for the changes in normal modes, correspond exactly to the results originally given by Rayleigh in Ref. 2 for first-order perturbation theory. To see this, it is only necessary to substitute in his Eqs. (21) and (30) for the nth generalized stiffness K_n, the equivalent terms $\omega_n^2 M_n^x$, where ω_n is the natural frequency of the nth normal mode and M_n is the corresponding generalized mass. Rayleigh, in the same work, also gives the complete development of second-order perturbation theory.

The only difference between the usual form of first-order perturbation theory applied to vibration problems and Stetson's results is that, following Rayleigh for undamped systems, it is customary to calculate the perturbation in frequency squared, $\Delta(\omega_n^2)$. Whereas Stetson computes $\Delta\omega_n$ so that $\Delta(\omega^2)$ is given by $2\omega_{no}\Delta\omega_n$ where ω_{no} is the unperturbed natural frequency. Formally, to mathematical terms of the first order, these expressions are equivalent. In practice it appears to be neither necessary nor desirable to make this substitution. For example, if the stiffness of a structure is uniformly increased 10% (say by changing materials), the change in ω_n^2 is exactly 10%. The new values of ω_n are exactly $1.1\omega_{no}$ or 4.88% higher than the original values. However, using Stetson's formulas, we get a change of 5%. If the new value of ω_n is taken as $1.05\omega_{no}$, then its square ω_n^2 rather than $1.1\omega_{no}^2$. Although the differences are minor as long as the perturbation parameters are small, there is no reason to change long-established methods to introduce additional errors, no matter how small. Moreover, in certain instances, taking advantage of the variational properties of some perturbation formulas, they are used when perturbations are not small, and in such cases unnecessary errors should certainly be avoided.

Finally, it is worth nothing with regard to the applications of perturbation theory alluded to by Stetson that Rayleigh[2] has specifically discussed the problem of "tailoring" perturbations so as to produce given changes in the natural frequencies of vibrating systems using the vibrating string as an example. He has also shown how mass distribution to achieve desired frequencies can be calculated on a basis more precise than first-order perturbation theory.

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Index categories: Structural Design, Optimal; Structural Dynamic Analysis.

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Reply by Author to A.H. Flax

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THE comments of A.H. Flax are well taken. I did not reference any prior perturbation analyses because I felt there were simply too many, and none that I knew of were really suited to the task I wished to address. The one by Lord Rayleigh is certainly the closest to what I was seeking; however, it is overly concise, occupying a scant two pages, and the illustrations using the string are too simple because only mass perturbations are used. Without demeaning Rayleigh's work, which is truly monumental and of which the perturbation analysis is but one of hundreds of topics, I would like to state what I found lacking. First, mode shapes are not explicitly used in the formulation. Their influence appears implicitly in two parameters, a_r and c_r, which correspond to modal mass and modal stiffness, and which are never explicitly defined except later in the book in various contexts. Second, he never actually stated that the new mode shape is expressible as a series of the old mode shapes, certainly not mathematically. Third, the physical interpretation of mode shape changes in not discussed at all, that is, Rayleigh confined himself to physical interpretations of frequency changes.

The primary point of my article is supported by Rayleigh's introduction to this topic. He presented perturbation as a method of analyzing a complicated system in terms of a simple one for which an analysis is mathematically known. My point is that the experimental technique of hologram interferometry allows even very complicated structures to be taken as starting points for perturbation analysis.

Finally, with regard to applications of perturbation theory to design, a colleague and I have currently under submission to this journal an article in the inverse of the perturbation process to specify structural changes that will change mode shapes and frequencies deterministically. We have formulated this both from a finite element and a continuum point of view. We look forward to the comments of Dr. Flax on this topic.

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